Mortgage Loan Derivation

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Abstract

We derive the equations for a mortgage loan and give several useful formulas.

1 The Repayment Structure of a Mortgage Loan

1.1 Definition of Interest Rate

The interest rate on the loan note is quoted as a yearly amount but is applied periodically for m periods per year, giving

$$i = \frac{Annual\,interest\,percent/100}{m} \tag{1}$$

Most house and car loans have a monthly period, $m = 12$.

NOTE: This is NOT the APR, which is the interest percentage as if the loan were compounded yearly instead of monthly. I would ignore the APR.

1.2 Periodic Payments of Principal + Interest

The j^{th} payment PMT consists of interest I_j on the previous unpaid balance BAL_{j-1}

This payment is a fixed amount every month by design.

Only the remainder of the payment after interest R_j actually reduces the principal of the loan.

Payments from the initial payment to the end payment have the form,

$$\begin{pmatrix} Payment\,Num & Payment\,Amount & Interest & Balance \\ 1^{st} & PMT = R_1 + I_1 & I_1 = i\,BAL_0 & BAL_1 = BAL_0 - R_1 \\ 2^{nd} & PMT = R_2 + I_2 & I_2 = i\,BAL_1 & BAL_2 = BAL_1 - R_2 \\ \ldots & \ldots & \ldots & \ldots \\ j^{th} & PMT = R_j + I_j & I_j = i\,BAL_{j-1} & BAL_j = BAL_{j-1} - R_j \\ \ldots & \ldots & \ldots & \ldots \\ n^{th} & PMT = R_n + I_n & I_n = i\,BAL_{n-1} & BAL_n = BAL_{n-1} - R_n = 0 \end{pmatrix} \tag{2}$$

n = the number of periods of the loan

The loan amount is $PV = BAL_0$ which is the original balance before any payments are made.

The last balance is 0 because the loan is paid off, $BAL_n = 0$.

We could write a Perl or Python script to compute this, but there are closed form solutions as we'll show.

2 Closed Form Solution

2.1 Difference Equation

Substituting we get a difference equation for the balances in terms of interest i and constant periodic payment PMT,

$$BAL_j = BAL_{j-1} - PMT + I_j \tag{3}$$

$$BAL_j = BAL_{j-1} - PMT + iBAL_{j-1} \tag{4}$$

$$BAL_j = (1+i)BAL_{j-1} - PMT \tag{5}$$

Let's solve this difference equation. But first, some lemmas,

Lemma The geometric progression

$$g = 1 + b + b^2 + \ldots + b^{n-1} \tag{6}$$

has the formula,

$$g = \frac{1-b^n}{1-b} \tag{7}$$

Proof

Multiply by b and subtract,

$$g - bg = g(1-b) = \left(1 + b + b^2 + \ldots + b^{n-1}\right) - \left(b + b^2 + b^3 + \ldots + b^n\right) = 1 - b^n \tag{8}$$

to get

$$g = \frac{1-b^n}{1-b} \tag{9}$$

QED.

Lemma. If

$$u_k = a + bu_{k-1} \tag{10}$$

then

$$u_n = a\frac{1-b^n}{1-b} + b^n u_0 \tag{11}$$

Proof. Expand out terms to see the pattern and use mathematical induction if you like,

$$u_1 = a + bu_0 \tag{12}$$

$$u_2 = a + bu_1 = a + b(a + bu_0) = a(1+b) + b^2 u_0 \tag{13}$$

$$u_3 = a + bu_2 = a + b[(1+b) + b^2 u_0] = a(1+b+b^2) + b^3 u_0 \tag{14}$$

The general formula is

$$u_n = a(1 + b + b^2 + \ldots + b^{n-1}) + b^n u_0 \tag{15}$$

Now use the geometric progression lemma above.
QED.

2.2 Closed Form Solution for BAL_k

To solve the balance difference equation let $u_k = BAL_k$, $a = -PMT$, and $b = i + 1$

The balance AFTER payment k is then

$$BAL_k = (-PMT)\frac{1-(1+i)^k}{1-(1+i)} + (1+i)^k BAL_0 \tag{16}$$

rearranging,

$$BAL_k = \frac{1}{(1+i)^{-k}}\left(PV + PMT\frac{(1+i)^{-k}-1}{i}\right) \tag{17}$$

Now set $BAL_n = 0$ because the last nth payment PMT finishes the mortgage.
Check: $BAL_0 = PV$

3 Other Useful Closed Form Solutions

3.1 Periodic Payment PMT From PV, n and i

The constant periodic payment is then

$$PMT = \frac{iPV}{1-(1+i)^{-n}} \tag{18}$$

3.2 Number of Periods n from PV, i and PMT

The number of periods (months) comes from the loan amount PV and the periodic payment PMT after inverting,

$$n = -\frac{ln\left(1 - i\frac{PV}{PMT}\right)}{ln(1+i)} \tag{19}$$

where we're using the natural logarithm, but any log will work.

3.3 Reduced Number of Periods n' from Increasing the PMT

You can use this to determine the reduced number of periods if you prepay principal. Change PMT to $(PMT + additional\ principal\ prepayment)$

$$n' = -\frac{ln\left(1 - i\frac{PV}{PMT + additional\ principal\ prepayment}\right)}{ln(1+i)} \tag{20}$$

3.4 Equivalent Simple Interest i'

Equivalent simple interest i' on the loan is how much interest would have been had it been payable up front immediately.

We use this formula to deduce it,

$$PV + i'PV = nPMT \tag{21}$$

Equivalent simple interest is then

$$i' = \frac{ni}{1-(1+i)^{-n}} - 1 \tag{22}$$

3.5 Accumulated Interest Payments j though k

What is the accumulated interest for payments j though k inclusive?

$$Int_{j \to k} = I_j + \ldots + I_k \tag{23}$$

$$Int_{j \to k} = (k - j + 1)\, PMT - \sum_{l=j}^{k} R_l \tag{24}$$

but

$$BAL_l - BAL_{l-1} = -R_l \tag{25}$$

We get a telescoping sum

$$-\sum_{l=j}^{k} R_l = BAL_k - BAL_{j-1} \tag{26}$$

Interest AFTER the kth payment is

$$Int_{1\to k} = kPMT + BAL_k - PV \tag{27}$$

Check: When $k = n$,

$$Int_{1\to n} = nPMT + BAL_n - PV = nPMT - PV \tag{28}$$

as expected because the total n payments of PMT include repaying PV plus interest.